KJMO 2015

Day 1

1. In an acute, scalene triangle $\triangle ABC$, let O be the circumcenter. Let M be the midpoint of AC. Let the perpendicular from A to BC be D. Let the circumcircle of $\triangle OAM$ hit DM at $P(\neq M)$. Prove that B, O, P are collinear.

2. For a positive integer m, prove that the number of pairs of positive integers (x, y) which satisfies the following two conditions is even or 0.

   (i) $x^2 - 3y^2 + 2 = 16m$

   (ii) $2y \leq x - 1$

3. For all nonnegative integer i, there are seven cards with 2^i written on it. How many ways are there to select the cards so that the numbers add up to n?

4. Reals a, b, c, x, y satisfy $a^2 + b^2 + c^2 = x^2 + y^2 = 1$. Find the maximum value of

$$(ax + by)^2 + (bx + cy)^2$$

Day 2

5. Let I be the incenter of an acute triangle $\triangle ABC$, and let the incircle be Γ. Let the circumcircle of $\triangle IBC$ hit Γ at D, E, where D is closer to B and E is closer to C. Let $\Gamma \cap BE = K(\neq E)$, $CD \cap BI = T$, and $CD \cap \Gamma = L(\neq D)$. Let the line passing T and perpendicular to BI meet Γ at P, where P is inside $\triangle IBC$. Prove that the tangent to Γ at P, KL, BI are concurrent.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

   (i) For different reals x, y, $f(x) \neq f(y)$.

   (ii) For all reals x, y, $f(x + f(f(-y))) = f(x) + f(f(y))$

7. For a polynomial $f(x)$ with integer coefficients and degree no less than 1, prove that there are infinitely many primes p which satisfies the following.

   There exists an integer n such that $f(n) \neq 0$ and $|f(n)|$ is a multiple of p.

8. A positive integer n is given. If there exist sets $F_1, F_2, \ldots, F_m$ satisfying the following, prove that $m \leq n$. (For sets A, B, $|A|$ is the number of elements in A. $A - B$ is the set of elements that are in A but not B)

   (i) For all $1 \leq i \leq m$, $F_i \subseteq \{1, 2, \ldots, n\}$

   (ii) $|F_1| \leq |F_2| \leq \ldots, \leq |F_m|$

   (iii) For all $1 \leq i < j \leq m$, $|F_i - F_j| = 1$.